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Superintegrability and higher order polynomial algebras

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Abstract
We present a method to obtain higher order integrals and polynomial algebras for two-dimensional quantum superintegrable systems separable in Cartesian coordinates from ladder operators. All systems with a second- and a third-order integral of motion separable in Cartesian coordinates were studied. The integrals of motion of two of them do not generate a cubic algebra. We construct for these Hamiltonians a higher order polynomial algebra from their ladder operators. We obtain quintic and seventh-order polynomial algebras. We also give for the polynomial algebras of order 7 realizations in terms of deformed oscillator algebras. These realizations and finite-dimensional unitary representations allow us to obtain the energy spectrum. We also apply the construction to the caged anisotropic harmonic oscillator and a system involving the fourth Painlevé transcendent.

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1. Introduction

Over the years many articles were devoted to superintegrability [1–23]. We recall that in classical mechanics a Hamiltonian system with Hamiltonian H and integrals of motion X_a

$$H = \tfrac{1}{2} g_{ik} p_i p_k + V(\vec{x}, \vec{p}), \qquad X_a = f_a(\vec{x}, \vec{p}), \qquad a = 1, \ldots, n-1, \tag{1.1}$$

is called completely integrable (or Liouville integrable) if it allows n integrals of motion (including the Hamiltonian), that are well-defined functions on phase space, are in involution $\{H, X_a\}_p = 0$, $\{X_a, X_b\}_p = 0$, $a, b = 1, \ldots, n-1$ and are functionally independent ($\{,\}_p$ is a Poisson bracket). A system is superintegrable if it is integrable and allows further integrals of motion $Y_b(\vec{x}, \vec{p})$, $\{H, Y_b\}_p = 0$, $b = 1, \ldots, k$, that are also well-defined functions on phase space and the integrals $\{H, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k\}$ are functionally independent. A system is maximally superintegrable if the set contains $(2n-1)$ such integrals. The integrals Y_b are not required to be in involution with $X_1, \ldots, X_{n-1}$, nor with each other. The same definitions apply in quantum mechanics but $\{H, X_a, Y_b\}$ are well-defined quantum mechanical operators and are assumed to form an algebraically independent set. However, this definition ignores more general relations between integrals. Finding an appropriate and rigorous definition of the independence of quantum operators is a difficult task and there is no agreed definition of quantum functional independence.

Superintegrable systems in classical and quantum mechanics possess many properties and appear to be important from the point of view of mathematics and physics. A review of two-dimensional classical and quantum superintegrable systems and their properties was made in a recent article [13]. Their non-Abelian algebraic structure generated by their integrals of motion can be a Lie algebra [1–3], a Kac–Moody algebra [20], a quadratic algebra [12, 21–23] or a cubic algebra [13, 17, 18]. These polynomial algebras were related to the deformed oscillator and parafermionic algebras [23] and can be used to obtain algebraically the degenerate energy spectrum.

Superintegrable systems are also related to systems studied in supersymmetric quantum mechanics (SUSYQM) [25–30] and higher order supersymmetric quantum mechanics (HSQM) [31–36]. In recent articles, we discussed how systems with a second- and third-order integral of motion are related to SUSYQM [17] and HSQM [18]. Supersymmetry can also be used to generate quantum superintegrable systems with higher order integrals of motion [19]. For well-known superintegrable systems such as the isotropic and the anisotropic harmonic oscillator, the Kepler–Coulomb and Smorodinsky–Winternitz systems, the relation between integrals of motion, ladder operators and supercharges have been discussed [3, 6, 8, 37–43]. The relation between the integrals of motion of superintegrable systems and intertwining operators was also discussed [44–46]. More recently, the integrals of motion of finite-gap systems and reflectionless Pöschl–Teller were also related to supercharges in hidden bozonized nonlinear supersymmetry [47, 48]. In the light of these results, it is interesting to pose the following question: Can we obtain a connection between the integrals of motion, supercharges and ladder operators for other quantum systems and in particular for two-dimensional superintegrable systems with higher order integrals of motion?

In the two-dimensional Euclidean space E_2 there are eight classical and fourteen quantum systems with a second- and a third-order integral [13, 15, 17, 18]. The quantum systems were investigated from the point of view of cubic algebras and supersymmetric quantum mechanics [17, 18]. We obtained for these systems the supercharges, the wavefunctions and the energy spectrum. For two of these quantum superintegrable systems, the integrals do not generate a cubic algebra [17] but we constructed the ladder operators from the supercharges. The purpose of this paper is to obtain the integrals and the polynomial algebra from ladder operators for these systems but also for a certain class of superintegrable systems allowing the separation of variables in Cartesian coordinates.

Let us present the organization of this paper. In section 2, we present a method to generate higher order integrals and a polynomial algebra for two-dimensional Hamiltonians separable in Cartesian coordinates constructed from two one-dimensional Hamiltonians and their ladder operators. In section 3, we apply the results of section 2 to the Smorodinsky–Winternitz potentials and three systems with a second- and a third-order integral of motion (potentials 1, 5 and 6 of [17]). In section 4, we give for a class of polynomial algebras of order 7 the realizations in terms of deformed oscillator algebras. In section 5, we use the results of sections 3 and 4 to obtain the Fock-type unitary representations and the corresponding energy spectrum of the potentials 5 and 6. In section 6, we apply the construction to the caged anisotropic harmonic oscillator and construct a new system involving the fourth Painlevé transcendent.

2. Polynomial algebras

Let us consider a two-dimensional Hamiltonian separable in Cartesian coordinates

$$H(x, y, P_x, P_y) = H_x(x, P_x) + H_y(y, P_y), \tag{2.1}$$

for which the ladder operators A_x, $A_x^\dagger$, A_y and $A_y^\dagger$ (polynomials in momenta) exist and satisfy relations of deformed oscillator algebras [49] or polynomial Heisenberg algebras [34]:

$$\left[H_x, A_x^\dagger\right] = \lambda_x A_x^\dagger, \quad [H_x, A_x] = -\lambda_x A_x, \quad A_x A_x^\dagger = Q(H_x + \lambda_x), \quad A_x^\dagger A_x = Q(H_x), \tag{2.2}$$

$$[H_y, A_y^\dagger] = \lambda_y A_y^\dagger, \quad [H_y, A_y] = -\lambda_y A_y, \quad A_y, A_y^\dagger = S(H_y + \lambda_y), \quad A_y^\dagger A_y = S(H_y). \tag{2.3}$$

These relations can also be interpreted as polynomial superalgebras [31–37, 50], i.e. $\left\{A_x, A_x^\dagger\right\} = Q(H_x + \lambda_x) + Q(H_x)$. Many well-known one-dimensional quantum systems possess ladder operators satisfying such algebraic structures [24–36].

The operators

$$f_1 = A_x^{\dagger m} A_y^n, \qquad f_2 = A_x^m A_y^{\dagger n} \tag{2.4}$$

commute with the Hamiltonian H given by equation (2.1)

$$[H, f_1] = [H, f_2] = 0, \tag{2.5}$$

if

$$m\lambda_x - n\lambda_y = 0, \qquad m, n \in \mathbb{Z}^+. \tag{2.6}$$

The ladder operators allow us to construct polynomial integrals of motion. The following sums are also polynomial integrals that commute with the Hamiltonian H:

$$I_1 = A_x^{\dagger m} A_y^n - A_x^m A_y^{\dagger n}, \qquad I_2 = A_x^{\dagger m} A_y^n + A_x^m A_y^{\dagger n}. \tag{2.7}$$

The order of these integrals of motion depends on the order of the ladder operators. The separation of variable in Cartesian coordinates implies the existence of a second-order integral $A = H_x - H_y$ [6]. The two-dimensional system given by equation (2.1) is thus superintegrable. The ladder operators satisfy polynomial Heisenberg algebras (or polynomial superalgebras) and can also provide a method to determine the polynomial algebra generated by the integrals of motion of the two-dimensional superintegrable systems given by equation (2.1). Let us now discuss two cases.

2.1. Case $\lambda_x = \lambda_y = \lambda$

We take the following linear combination:

$$A = 2(H_x - H_y), \qquad I_1 = \left(A_x^\dagger A_y - A_x A_y^\dagger\right), \qquad I_2 = 4\lambda\left(A_x^\dagger A_y + A_x A_y^\dagger\right). \tag{2.8}$$

We use the integrals given by equation (2.9) to construct polynomial algebras of the Hamiltonian given by equation (2.1).

$$\begin{aligned} &[A, I_1] = I_2, \quad [A, I_2] = 16\lambda^2 I_1, \\ &[I_1, I_2] = 8\lambda(Q\left(\tfrac{1}{2}\left(H + \tfrac{1}{2}A\right)\right) S\left(\tfrac{1}{2}\left(H - \tfrac{1}{2}A\right) + \lambda\right) \\ &\qquad - Q\left(\tfrac{1}{2}\left(H + \tfrac{1}{2}A\right) + \lambda\right) S\left(\tfrac{1}{2}\left(H - \tfrac{1}{2}A\right)\right). \end{aligned} \tag{2.9}$$

2.2. Case $2\lambda_x = \lambda_y = \lambda$

We take the following integrals:

$$A = 2(H_x - H_y), \qquad I_1 = \left(A_x^{\dagger 2} A_y - A_x^2 A_y^\dagger\right), \qquad I_2 = 4\lambda\left(A_x^{\dagger 2} A_y + A_x^2 A_y^\dagger\right). \tag{2.10}$$

The polynomial algebra is thus

$$\begin{aligned} &[A, I_1] = I_2, \qquad [A, I_2] = 16\lambda^2 I_1, \\ &[I_1, I_2] = 8\lambda\big(Q\left(\tfrac{1}{2}\left(H + \tfrac{1}{2}A\right) - \lambda_x\right) Q\left(\tfrac{1}{2}(H + \tfrac{1}{2}A)\right) S\left(\tfrac{1}{2}\left(H - \tfrac{1}{2}A\right) + \lambda_y\right) \\ &\qquad - Q\left(\tfrac{1}{2}\left(H + \tfrac{1}{2}A\right) + 2\lambda_x\right) Q\left(\tfrac{1}{2}\left(H + \tfrac{1}{2}A\right) + \lambda_x\right) S\left(\tfrac{1}{2}\left(H - \tfrac{1}{2}A\right)\right)\big). \end{aligned} \tag{2.11}$$

For these two cases, the order of this polynomial algebra depends on the order of the polynomials $Q(H_x)$ and $S(H_y)$. Examples of such construction were used to write the angular momentum algebra as two independent harmonic oscillators [51] and obtain quadratic algebras from Lie algebras [39]. We will present the general case in section 6.

3. Applications

There are two Smorodinsky–Winternitz potentials [6] that allow separation of variables of the Schrödinger equation in Cartesian coordinates: $V_a(x, y) = \frac{\omega}{2}(x^2 + y^2) + \frac{b}{x^2} + \frac{c}{y^2}$ and $V_b(x, y) = \frac{\omega^2}{2}(4x^2 + y^2) + bx + \frac{c}{y^2}$. They are well-known quadratically superintegrable systems and we apply the construction to these two systems. The potential V_a has in the x axis the following ladder operators:

$$A_x^\dagger = -\frac{1}{4}\left(\frac{\hbar}{\omega}\frac{d^2}{dx^2} - 2x\frac{d}{dx} + \frac{\omega}{\hbar}x^2 - \frac{2b}{\omega\hbar x^2} - 1\right), \tag{3.1}$$

$$A_x = -\frac{1}{4}\left(\frac{\hbar}{\omega}\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{\omega}{\hbar}x^2 - \frac{2b}{\omega\hbar x^2} + 1\right). \tag{3.2}$$

These operators were first obtained in [6] and reobtained in a systematic study of systems with second-order ladder operators [52]. In the y axis there are ladder operators of the same form. The polynomial $Q(H_x)$ is given by

$$Q(H_x) = \frac{1}{4\hbar^2\omega^2}H_x^2 - \frac{1}{2\hbar\omega}H_x + \left(\frac{3}{16} - \frac{b}{2\hbar^2}\right). \tag{3.3}$$

The polynomial $S(H_y)$ is also given by equation (3.3) (by replacing H_x by H_y and b by c). We can form with $Q(H_x)$, $S(H_y)$, equations (2.8) and (2.9) the integrals and the polynomial algebra. This polynomial algebra can also be determined for V_b. In the two cases the polynomial algebra is a cubic algebra where the generators are second-, third- and fourth-order operators. However, we can form a simpler algebraic structure for these Hamiltonians, a quadratic algebra where the generators are two second-order and one third-order operators [16]. The integrals obtained from the construction of section 2 are not necessarily the integrals of the lowest possible order.

3.1. Systems with a third-order integral

We considered well-known quadratically superintegrable systems. We will apply the method to potentials 1, 5 and 6, and present their integrals and polynomial algebras. The first system that we consider is potential 1 [15, 17]:

$$H = \frac{1}{2}P_x^2 + \frac{1}{2}P_y^2 + \hbar^2\left(\frac{x^2 + y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2}\right). \tag{3.4}$$

In the x axis, the ladder operators were constructed from supersymmetry with the supercharge [17] and are given by

$$A_x = \frac{\hbar^2}{4a^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2a^2}x + \left(\frac{1}{x-a} + \frac{1}{x+a}\right)\right)\left(x + 2a^2\frac{\mathrm{d}}{\mathrm{d}x}\right) \times \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2a^2}x + \left(\frac{1}{x-a} + \frac{1}{x+a}\right)\right), \tag{3.5}$$

$$A_x^\dagger = \frac{\hbar^2}{4a^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2a^2}x + \left(\frac{1}{x-a} + \frac{1}{x+a}\right)\right)\left(x - 2a^2\frac{\mathrm{d}}{\mathrm{d}x}\right) \times \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2a^2}x + \left(\frac{1}{x-a} + \frac{1}{x+a}\right)\right), \tag{3.6}$$

$$A_y = \frac{\hbar}{2a^2}\left(y + 2a^2\frac{\mathrm{d}}{\mathrm{d}y}\right), \qquad A_y^\dagger = \frac{\hbar}{2a^2}\left(y - 2a^2\frac{\mathrm{d}}{\mathrm{d}y}\right). \tag{3.7}$$

We construct the known integral B of order 3 [15, 17] from equation (2.8) $\left(B = I_1' = \frac{-2a^2\mathrm{i}}{\hbar}I_1\right)$. We have $\lambda = \frac{\hbar^2}{2a^2}$. We have also presented the commutator of these ladder operators in [17]. In the x and y axes the polynomial Heisenberg algebras are given by equations (2.2) and (2.3) with the following expressions:

$$Q(H_x) = 2H_x^3 - \frac{7}{2}\frac{\hbar^2}{a^2}H_x^2 + \frac{7\hbar^4}{8a^4}H_x + \frac{15\hbar^6}{32a^6} \quad S(H_y) = 2H_y - \frac{\hbar^2}{2a^2}. \tag{3.8}$$

We get from equation (2.9) of the previous section and equation (3.8) the following cubic algebra that coincides with the one found in [17].

$$[A, I_1'] = I_2', \qquad [A, I_2'] = \frac{4h^4}{a^4}I_1', \qquad [I_1', I_2'] = -2\hbar^2A^3 - 6\hbar^2A^2H + 8\hbar^2H^3 + 6\frac{\hbar^4}{a^2}A^2 + 8\frac{\hbar^4}{a^2}HA - 8\frac{\hbar^4}{a^2}H^2 + 2\frac{\hbar^6}{a^4}A - 2\frac{\hbar^6}{a^4}H - 6\frac{\hbar^8}{a^6}.$$

The energy spectrum was calculated from the Fock-type unitary representations [17]. In section 4, we will extend this algebraic method of calculating the energy spectrum of superintegrable systems with a polynomial algebra of order 7.

3.2. *Potential 6*

The next system that we consider is an Hamiltonian for which no polynomial algebra was found from the second- and third-order integrals of motion. We will show how the procedure of section 2 and the ladder operators obtained from supersymmetric quantum mechanics will allow us to find a quintic algebra. The Hamiltonian

$$H = \frac{1}{2}P_x^2 + \frac{1}{2}P_y^2 + \hbar^2\left(\frac{x^2+y^2}{8a^4} + \frac{1}{(x-a)^2} + \frac{1}{(x+a)^2} + \frac{1}{(y-a)^2} + \frac{1}{(y+a)^2}\right) \tag{3.9}$$

has the following second-order integral $A = H_x - H_y$ and third-order integral:

$$B = 2L^3 - 3\alpha^2(\{L, P_x^2\} + \{L, P_y^2\}) + \frac{\hbar^2}{4}\left\{(124y + 3\left(\frac{y}{a^2}\right)(x^2+y^2) + 24y\frac{(x^2-5y^2)}{(y^2-a^2)} - \frac{144yx^2}{x^2-a^2} + 24y\frac{(3x^2-y^2)(x^2+a^2)}{(x^2-a^2)^2} + 48y\frac{(y^2-x^2)(y^2+a^2)}{(y^2-a^2)}, P_x\right\}$$

$$-\frac{\hbar^2}{4}\left\{(124x+3\left(\frac{x}{a^2}\right)(y^2+x^2)+24x\frac{(y^2-5x^2)}{(x^2-a^2)}-\frac{144xy^2}{y^2-a^2}\right.$$
$$\left.+24x\frac{(3y^2-x^2)(y^2+a^2)}{(y^2-a^2)^2}+48x\frac{(x^2-y^2)(x^2+a^2)}{(x^2-a^2)},P_y\right\}. \tag{3.10}$$

The ladder operators are given by equations (3.5) and (3.6) in the x and y axes (by replacing x by y). The polynomial algebras in the x and y axes are given by equations (2.2) and (2.3) with $Q(H_x)$ given by equation (3.8) and $S(H_y)$ by the same expression (by replacing H_x by H_y). We have $\lambda=\frac{\hbar^2}{2a^2}$. The integrals of motion are given by equation (2.8) $\left(I_1'=\frac{-2a^2\mathrm{i}}{\hbar}I_1\right)$. Thus, we obtain with equation (2.9)

$$[A,I_1']=I_2',\qquad [A,I_2']=\frac{4\hbar^4}{a^4}I_1',\qquad [I_1',I_2']=-\frac{3}{16}\hbar^2A^5+\frac{3}{2}\hbar^2A^3H^2$$
$$-\frac{2\hbar^4}{a^2}A^3H-3\hbar^2AH^4+\frac{8\hbar^4}{a^2}AH^3+\frac{19\hbar^6}{8a^4}A^3-\frac{13\hbar^6}{2a^4}AH^2-\frac{99\hbar^{10}}{16a^8}A+\frac{6\hbar^8}{a^6}AH. \tag{3.11}$$

The integrals I_1' is related to the integrals A, B by $I_1'=\frac{-1}{384\hbar^2}[A,[A,B]]+\frac{3\hbar^2}{32a^4}B$.

3.3. *Potential 5*

The Hamiltonian

$$H=\frac{1}{2}P_x^2+\frac{1}{2}P_y^2+\hbar^2\left(\frac{x^2+y^2}{8a^4}+\frac{1}{(x+a)^2}+\frac{1}{(x-a)^2}+\frac{1}{y^2}\right) \tag{3.12}$$

has a quadratic $A=H_x-H_y$ and a cubic integral

$$B=2L^3-3a^2\{L,P_y\}+\hbar^2\left\{\frac{3}{4a^2}-\frac{6y^3(x^2+a^2)}{(x^2-a^2)^2}-\frac{3(x^2-a^2)}{y}-2y,P_x\right\},$$
$$3\hbar^2\left\{x(\frac{(x^2-3a^2)}{y^2}-\frac{(3y^2-8a^2)}{12a^2}-\frac{2y^2}{x^2-a^2}+\frac{4y^2(x^2+a^2)}{(x^2-a^2)^2},P_y\right\}. \tag{3.13}$$

The integrals *A*, *B* and their commutator do not generate a cubic algebra. We will construct other integrals of motion from the ladder operators. The ladder operators are given by equations (3.1), (3.2) (by replacing x by y and ω by a), (3.5) and (3.6). We have in the x and y axes polynomial algebras given by equations (2.2) and (2.3) with $\lambda=\frac{\hbar^2}{a^2}$ and

$$Q(H_x)=2H_x^3-\frac{7}{2}\frac{\hbar^2}{a^2}H_x^2+\frac{7\hbar^4}{8a^4}H_x+\frac{15\hbar^6}{32a^6},\qquad S(H_y)=\frac{a^4}{\hbar^4}H_y^2-\frac{a^2}{\hbar^2}H_y-\frac{5}{16}. \tag{3.14}$$

We obtain with equations (2.11) and (3.15) the following polynomial algebra with integrals given by equation (2.10) $(I_1'=a^2I_1)$:

$$[A, I_1'] = I_2', \quad [A, I_2'] = \frac{16\hbar^4}{a^4} I_1',$$

$$\begin{aligned}
[I_1', I_2'] = & \frac{75\hbar^{14}}{64a^{10}} - \frac{275H\hbar^{12}}{64a^8} - \frac{3H^2\hbar^{10}}{16a^6} + \frac{261H^3\hbar^8}{16a^4} - \frac{75H^4\hbar^6}{4a^2} + \frac{15H^5\hbar^4}{4} \\
& + 3a^2H^6\hbar^2 - \frac{a^4A^7}{64} - a^4H^7 + A^6\left(\frac{7a^2\hbar^2}{64} - \frac{7a^4H}{64}\right) \\
& + A^5\left(-\frac{3}{16}H^2a^4 + \frac{9}{16}H\hbar^2a^2 - \frac{25\hbar^4}{64}\right) \\
& + A^4\left(\frac{45\hbar^6}{64a^2} - \frac{85H\hbar^4}{64} + \frac{5}{16}a^2H^2\hbar^2 + \frac{5a^4H^3}{16}\right) \\
& + A^3\left(\frac{21\hbar^8}{64a^4} + \frac{5H\hbar^6}{8a^2} + \frac{15H^2\hbar^4}{8} - \frac{5}{2}a^2H^3\hbar^2 + \frac{5a^4H^4}{4}\right) \\
& + A^2\left(-\frac{127\hbar^{10}}{64a^6} + \frac{239H\hbar^8}{64a^4} - \frac{85H^2\hbar^6}{8a^2} + \frac{95H^3\hbar^4}{8} - \frac{15}{4}a^2H^4\hbar^2 + \frac{3a^4H^5}{4}\right) \\
& + A\left(\frac{5\hbar^{12}}{64a^8} - \frac{35H\hbar^{10}}{16a^6} + \frac{229H^2\hbar^8}{16a^4} - \frac{55H^3\hbar^6}{2a^2} + \frac{55H^4\hbar^4}{4} + a^2H^5\hbar^2 - a^4H^6\right).
\end{aligned} \tag{3.15}$$

The integral I_1' is of order 7 and the integral I_2' is of order 8.

4. Realizations of polynomial algebras

In the previous section, we generated polynomial algebras of many systems. These algebras were cubic-, quintic- and seventh-order algebras. In earlier articles it was demonstrated that the quadratic [23] and cubic [17] algebras can be realized as deformed oscillator algebras [49] that allow us to construct Fock-type representations and obtain the energy spectrum. We will show that we can construct similar realizations for the following polynomial algebra of order 7:

$$\begin{aligned}
&[A, B] = C, \qquad [A, C] = \delta B, \\
&[B, C] = mA^7 + nA^6 + \mu A^5 + \nu A^4 + \alpha A^3 + \beta A^2 + \gamma A + \epsilon,
\end{aligned} \tag{4.1}$$

where A and B are integrals and thus commute with the Hamiltonian H. The structure constants m, n, μ, ν, α, β, γ and ϵ are polynomials of the Hamiltonian. We do not impose an order to these integrals and only make the hypothesis that they generate an algebra of the form given by equation (4.1). The Casimir operator satisfies $[K, A] = [K, B] = [K, C] = 0$, and this implies

$$\begin{aligned}
K = & C^2 - \delta B^2 + \frac{m}{4}A^8 + \frac{2}{7}nA^7 + \left(\frac{\mu}{3} + \frac{7}{6}\delta m\right)A^6 + \left(\frac{2}{5}\nu + \delta n\right)A^5 \\
& + \left(\frac{\alpha}{2} + \frac{5}{6}\delta\mu - \frac{7}{12}\delta^2 m\right)A^4 + \left(\frac{2}{3}\beta + \frac{2}{3}\delta\nu - \frac{1}{3}\delta^2 n\right)A^3 \\
& + \left(\frac{\delta\alpha}{2} - \frac{1}{6}\delta^2\mu + \gamma + \frac{1}{6}\delta^3 m\right)A^2 + \left(2\epsilon + \frac{1}{3}\delta\beta + \frac{1}{21}\delta^3 n - \frac{\delta^2\nu}{15}\right)A.
\end{aligned} \tag{4.2}$$

The order of the Casimir operator depends on the order of A and B. Ultimately, the Casimir operator is written in terms of the Hamiltonian. There is a realization in terms of deformed oscillator algebras of the form

$$A = \delta(N + u), \qquad B = b^\dagger + b, \tag{4.3}$$

where u is an arbitrary constant and $\{N, b, b^\dagger\}$ satisfy

$$[N,b]=-b, \qquad [N,b^\dagger]=b^\dagger, \qquad bb^\dagger=\Phi(N+1), \qquad b^\dagger b=\Phi(N). \tag{4.4}$$

With the third relation of the seventh-order algebra given by equation (4.1) and the Casimir operator given by equation (4.3) we find

$$\begin{aligned}\Phi(N) = {} & \frac{m}{16}\delta^3(N+u)^8 + \left(\frac{n\delta^{\frac{5}{2}}}{14} - \frac{m\delta^3}{4}\right)(N+u)^7 + \left(\frac{\mu\delta^2}{12} + \frac{7}{24}m\delta^3 - \frac{n\delta^{\frac{5}{2}}}{4}\right)(N+u)^6 \\ & + \left(\frac{\nu\delta^{\frac{3}{2}}}{10} - \frac{\mu\delta^2}{4} + \frac{1}{4}n\delta^{\frac{5}{2}}\right)(N+u)^5 + \left(\frac{\alpha\delta}{8} + \frac{5\mu\delta^2}{24} - \frac{\nu\delta^{\frac{3}{2}}}{4} - \frac{7}{48}\delta^3 m\right)(N+u)^4 \\ & + \left(\frac{\beta\delta^{\frac{1}{2}}}{6} + \frac{\nu\delta^{\frac{3}{2}}}{6} - \frac{\alpha\delta}{4} - \frac{1}{12}\delta^{\frac{5}{2}}n\right)(N+u)^3 \\ & + \left(\frac{\delta\alpha}{8} - \frac{\delta^2\mu}{24} + \frac{\gamma}{4} - \frac{\beta\delta^{\frac{1}{2}}}{4} + \frac{1}{24}\delta^3 m\right)(N+u)^2 \\ & + \left(\frac{\epsilon}{2\delta^{\frac{1}{2}}} + \frac{\delta^{\frac{1}{2}}\beta}{12} - \frac{\gamma}{4} - \frac{1}{84}\delta^{\frac{5}{2}}n - \frac{1}{60}\delta^{\frac{3}{2}}\nu\right)(N+u) - \frac{\epsilon}{4\delta^{\frac{1}{2}}} - \frac{K}{4\delta}.\end{aligned} \tag{4.5}$$

To obtain unitary representations we should impose three constraints on the structure function

$$\Phi(p+1,u_i,k)=0, \qquad \Phi(0,u,k)=0, \qquad \phi(x)>0, \qquad \forall \quad x>0. \tag{4.6}$$

5. Potentials 5 and 6

Equation (4.5) gives the structure function in terms of the parafermionic number N and the structure constants. The three conditions given by equation (4.6) provide a method to obtain the energy spectrum. From the results of sections 4 and 3, we can found the unitary representations and the corresponding energy spectrum for potentials 5 and 6.

5.1. Potential 6 and quintic algebras

The algebra of potential 6 is given by equation (3.12) is a particular case of the one given by equation (4.1). The structure constants are

$$\begin{aligned}&\delta=\frac{4\hbar^4}{a^4}, \qquad \mu=-\frac{3\hbar^2}{16}, \qquad \nu=\beta=\epsilon=0,\\ &\alpha=\frac{3}{2}\hbar^2H^2+\frac{2\hbar^4}{a^2}H+\frac{19\hbar^6}{8a^4}, \qquad \gamma=-3\hbar^2H^4+\frac{8\hbar^4}{a^2}H^3-\frac{13\hbar^6}{2a^4}H^2+\frac{6\hbar^8}{a^6}H-\frac{99\hbar^{10}}{16a^8}.\end{aligned} \tag{5.1}$$

This quintic algebra is generated by the integrals A, I_1' and I_2' respectively of orders 2, 5 and 6. We can write the Casimir operator given by equation (4.2) as a polynomial of the Hamiltonian only:

$$K=-4\hbar^2H^6+\frac{16\hbar^4}{a^2}H^5-\frac{5\hbar^6}{a^4}H^4-\frac{40\hbar^8}{a^6}H^3+\frac{141\hbar^{10}}{4a^8}H^2+\frac{9\hbar^{12}}{a^{10}}H-\frac{135\hbar^{14}}{16a^{12}}. \tag{5.2}$$

We can find with equation (4.6) the structure function and factorize it in the following way:

$$\Phi(x)=\frac{-\hbar^{10}}{4a^{8}}\left((x+u)-\left(\frac{a^2E}{\hbar^2}-\frac{3}{2}\right)\right)\left((x+u)-\left(\frac{-a^2E}{\hbar^2}-\frac{1}{2}\right)\right)$$
$$\left((x+u)-\left(\frac{a^2E}{\hbar^2}-\frac{1}{2}\right)\right)\left((x+u)-\left(\frac{-a^2E}{\hbar^2}+\frac{3}{2}\right)\right) \tag{5.3}$$
$$\times\left((x+u)-\left(\frac{a^2E}{\hbar^2}+\frac{3}{2}\right)\right)\left((x+u)-\left(\frac{-a^2E}{\hbar^2}+\frac{5}{2}\right)\right).$$

To obtain unitary representations we should impose three constraints given by equation (4.6). There are four solutions for $a=\mathrm{i}a_0,\ a_0\in\mathbb{R}$. Let us present these unitary representations with the corresponding constant u and energy spectrum:

Case: with $u_1=\frac{a^2E}{\hbar^2}+\frac{3}{2}$

$$E_1=\frac{\hbar^2(p+3)}{2a_0^2},\qquad \Phi_1(x)=\frac{\hbar^{10}}{4a_0^8}x(x+2)(x+3)(p+4-x)(p+3-x)(p+1-x), \tag{5.4}$$

Case: with $u_2=\frac{a^2E}{\hbar^2}-\frac{1}{2}$ $(p=0{,}1)$

$$E_2=\frac{\hbar^2(p+1)}{2a_0^2},\qquad \Phi_3(x)=\frac{\hbar^{10}}{4a_0^8}x(x-2)(p+4-x)(p+3-x)(p+1-x), \tag{5.5}$$

Case: with $u_3=\frac{a^2E}{\hbar^2}-\frac{3}{2}$ ($p=0,1,2$ and $p=0$ respectively)

$$E_{3a}=\frac{\hbar^2(p)}{2a_0^2},\qquad \Phi_{3a}(x)=\frac{\hbar^{10}}{4a_0^8}x(x-3)(p+1-x)(p+3-x)(p+4-x), \tag{5.6}$$

$$E_{3b}=\frac{\hbar^2(p-3)}{2a_0^2},\qquad \Phi_{3b}(x)=\frac{\hbar^{10}}{4a_0^8}x(x-3)(p+1-x)(p-2-x)(p-x). \tag{5.7}$$

We must also exclude spurious states with other conditions. One of them consists in $E\geqslant \min V$. We have unitary representations valid only for $p=0$, $p=0$, 1 and $p=0$, 1, 2. Such solutions were also found in the context of cubic algebras. This phenomenon is related to zero modes, singlet state, doublet states and higher order supersymmetric quantum mechanics [17, 18]. A singlet state is annihilated by the creation operator. The energy spectrum is confirmed by the results obtained from supersymmetric quantum mechanics.

There is one solution for the case $a\in\mathbb{R}$ with $u=\frac{-a^2E}{\hbar^2}+\frac{5}{2}$. The unitary representation is (with $p\geqslant 3$)

$$E_1=\frac{\hbar^2(p+5)}{2a^2},\qquad \Phi_1(x)=\frac{\hbar^{10}}{4a^8}x(p+1-x)(x+3)(p+4-x)(p+2-x). \tag{5.8}$$

5.2. Potential 5 and polynomial algebras of order 7

The algebra of potential 5 is given by (3.15). This case belongs to the one given by equation (4.1). The structure constants are obtained by comparing equations (3.15) and (4.1). This seventh-order algebra is generated by the integrals A, I_1' and I_2' respectively of orders 2, 7 and 8. The Casimir operator given by equation (4.2) can be written as a function of the Hamiltonian only:

$$K = a^4H^8 - 4a^2\hbar^2H^7 + 3\hbar^4H^6 + \frac{15\hbar^6}{a^2}H^5 - \frac{453\hbar^8}{8a^4}H^4 + \frac{261\hbar^{10}}{4a^6}H^3 - \frac{133\hbar^{12}}{16a^8}H^2 - \frac{275\hbar^{14}}{16a^{10}}H + \frac{1425\hbar^{16}}{256a^{12}}. \quad (5.9)$$

Equation (4.6) give us the structure function and we can factorize it in the following way:

$$\Phi(x) = \left(\frac{4\hbar^{12}}{a^8}\right)\left(x+u-\left(-\frac{1}{4}-\frac{a^2E}{2\hbar^2}\right)\right)\left(x+u-\left(-\frac{1}{4}+\frac{a^2E}{2\hbar^2}\right)\right)$$
$$\times\left(x+u-\left(\frac{1}{4}-\frac{a^2E}{2\hbar^2}\right)\right)\left(x+u-\left(\frac{3}{4}-\frac{a^2E}{2\hbar^2}\right)\right)\left(x+u-\left(\frac{5}{4}-\frac{a^2E}{2\hbar^2}\right)\right)$$
$$\times\left(x+u-\left(\frac{5}{4}-\frac{a^2E}{2\hbar^2}\right)\right)\left(x+u-\left(\frac{5}{4}+\frac{a^2E}{2\hbar^2}\right)\right)\left(x+u-\left(\frac{7}{4}-\frac{a^2E}{2\hbar^2}\right)\right). \quad (5.10)$$

Let us present the solutions for the case $a = a_0 \mathrm{i}, a_0 \in \mathbb{R}$. There are two solutions for $u = \frac{5}{4} + \frac{a^2E}{2\hbar^2}$:

$$E_1 = \frac{\hbar^2(5+2p)}{2a_0^2}, \qquad \Phi_1(x) = \left(\frac{\hbar^{12}}{4a_0^8}\right)x(p+1-x)(2p+3-2x)(2p+5-2x)^2$$
$$(p+2-x)(p+3-x)(3+2x). \quad (5.11)$$

$$E_2 = \frac{\hbar^2(1+p)}{a_0^2}, \qquad \Phi_2(x) = \left(\frac{\hbar^{12}}{4a_0^8}\right)(3+2p-2x)(5+2p-2x)^2$$
$$(p+1-x)(p+2-x)(p+3-x)(-3+2x), \qquad p = 0. \quad (5.12)$$

We confirmed these energy levels with the results obtained using the separability in Cartesian coordinates and the SUSYQM.

There is one solution for the case $a \in \mathbb{R}$. For $u = 2 - \frac{a^2E}{\hbar^2}$, we get

$$E_1 = \frac{\hbar^2(p+3)}{a^2}, \quad \Phi_1(x) = \left(\frac{\hbar^{12}}{4a^8}\right)x(p+1-x)\left(x+\frac{1}{2}\right)^2\left(x+\frac{3}{2}\right)(x+2)\left(p+\frac{5}{2}-x\right). \quad (5.13)$$

6. General case $m\lambda_x = n\lambda_y$

We have presented examples satisfying $\lambda_x = \lambda_y = \lambda$ and $2\lambda_x = \lambda_y = \lambda$. We will now discuss the general case. We will now consider integrals given by equation (2.4):

$$A = \frac{1}{2\lambda}(H_x - H_y), \qquad I_- = A_x^m A_y^{\dagger n}, \qquad I_+ = A_x^{\dagger m} A_y^n. \quad (6.1)$$

We obtain the polynomial algebra

$$[A, I_-] = -I_-, \qquad [A, I_+] = I_+, \qquad [I_-, I_+] = F_{m,n}(H, A+1) - F_{m,n}(H, A), \quad (6.2)$$

$$F_{m,n} = \prod_{i=1}^{m} Q\left(\frac{H}{2} + m\lambda_x A - (m-i)\lambda_x\right)\prod_{j}^{n} S\left(\frac{H}{2} - n\lambda_y A + j\lambda_y\right). \quad (6.3)$$

We can define

$$b^\dagger = I_+, \qquad b = I_-, \qquad N = A - u, \qquad \Phi(H, N) = F_{m,n}(H, N+u). \quad (6.4)$$

This algebra is thus a deformed oscillator algebra as given by equation (4.4).

6.1. Applications

6.1.1. Caged anisotropic harmonic oscillator. We consider the caged anisotropic harmonic oscillator [53, 54]

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{\omega^2}{2}(k^2x^2 + m^2y^2) + \frac{l_1}{x^2} + \frac{l_2}{y^2}, \tag{6.5}$$

where $m, k \in \mathbb{Z}^+$. The method of separation of variables allows us to solve the corresponding Schrödinger equation in terms of Laguerre polynomials [6] and to obtain the energy spectrum. However, the polynomial algebra remains to be determined. We apply to this system the construction of sections 2 and 6. Let us only show the creation operator in the x axis

$$A_x^\dagger = -\frac{1}{4}\left(\frac{\hbar}{\omega k}\frac{d^2}{dx^2} - 2x\frac{d}{dx} + \frac{\omega k}{\hbar}x^2 - \frac{2l_1}{\omega k\hbar x^2} - 1\right). \tag{6.6}$$

The ladder operators satisfy the relations given by equations (2.2), (2.3) and (2.6) with $\lambda_x = 2\hbar k\omega$, $\lambda_y = 2\hbar m\omega$ and $m\lambda_x = k\lambda_y$. We can apply the results of section 2. We have

$$Q(H_x) = \frac{1}{4\hbar^2k^2\omega^2}H_x^2 - \frac{1}{2\hbar k\omega}H_x + \left(\frac{3}{16} - \frac{l_1}{2\hbar^2}\right). \tag{6.7}$$

The integrals are given by equation (6.1). With equations (6.3) and (6.4) we obtain the following structure function:

$$\begin{aligned}\Phi_{m,k}(x) = m^2k^2\prod_{i=1}^{m}&\left(\frac{E}{4mk\hbar\omega} + x + u - 1 + \frac{i}{m} - \frac{1}{2m} - \frac{\nu_1}{2m}\right)\\ &\times\left(\frac{E}{4mk\hbar\omega} + x + u - 1 + \frac{i}{m} - \frac{1}{2m} + \frac{\nu_1}{2m}\right)\prod_{j=1}^{k}\\ &\times\left(\frac{E}{4mk\hbar\omega} - x - u + \frac{j}{k} - \frac{1}{2k} - \frac{\nu_2}{2k}\right)\left(\frac{E}{4mk\hbar\omega} - x - u + \frac{j}{k} - \frac{1}{2k} + \frac{\nu_2}{2k}\right),\\ \nu_1 = \sqrt{1 + \frac{8l_1}{\hbar^2}}, \qquad &\nu_2 = \sqrt{1 + \frac{8l_2}{\hbar^2}}.\end{aligned} \tag{6.8}$$

We should impose the constraints given by equation (2.17). We obtain the following solutions with $p = 1, 2, \ldots, m$, $q = 1, 2, \ldots, k$ and $N \in \mathbb{N}$:

$$\begin{aligned}u &= \frac{-E}{4mk\hbar\omega} + \frac{m-p}{m} + \frac{1}{2m} + \frac{\epsilon_1\nu_1}{2m},\\ E &= 2mk\hbar\omega\left(N + 2 + \frac{1 - 2p + \epsilon_1\nu_1}{2m} + \frac{1 - 2q + \epsilon_2\nu_2}{2k}\right),\end{aligned} \tag{6.9}$$

$$\begin{aligned}\Phi_{m,k}(x) = m^2k^2\prod_{i=1}^{m}&\left(x + \frac{i-p}{m}\right)\left(x + \frac{i-p}{m} + \frac{\epsilon_1\nu_1}{m}\right)\prod_{j=1}^{k}\\ &\times\left(N + 1 + \frac{j-q}{k} - x\right)\left(N + 1 + \frac{j-q}{k} + \frac{\epsilon_2\nu_2}{k}\right).\end{aligned} \tag{6.10}$$

6.1.2. System with Painlevé transcendent. In [15], five systems involving Painlevé transcendent [55] were found. One of these systems was written as a function of the fourth Painlevé transcendent:

$$H = \frac{P_x^2}{2} + \frac{P_y^2}{2} + g_1(x) + g_2(y), \tag{6.11}$$

$$g_1(x) = \epsilon_1 \frac{\hbar\omega_1}{2} f_1^{'}\left(\sqrt{\frac{\omega_1}{\hbar}}x\right) + \frac{\omega_1^2}{2}\left(x + \sqrt{\frac{\hbar}{\omega_1}} f_1\left(\sqrt{\frac{\omega_1}{\hbar}}x\right)\right)^2 + \frac{\hbar\omega_1}{3}(-\alpha_1 + \epsilon_1), \tag{6.12}$$

$$g_2(y) = \frac{\omega_1^2}{2} y^2. \tag{6.13}$$

We presented its cubic algebra, wavefunctions and ladder operators [18]. The Hamiltonians with $\epsilon = 1$ and $\epsilon = -1$ are superpartners. They are also related to a special case of third-order supersymmetry called the shape invariance [33]. The function $f_1 = f_1\left(\sqrt{\frac{\omega_1}{\hbar}}x, \alpha_1, \beta_1\right)$ is the fourth Painlevé transcendent. The third-order ladder operators were also discussed in [30, 36]. The polynomial Heisenberg algebra of these operators was also obtained. They satisfy the relation given by equations (2.2) and (2.3) with $\lambda_x = \hbar\omega_1$ and

$$Q(H_x) = 8\left(H_x - \frac{\hbar\omega}{3}(-\alpha_1 + \epsilon_1 + 3)\right)\left(\left(H_x - \frac{\hbar\omega}{3}\left(\frac{\alpha_1}{2} + 4\epsilon_1 - \frac{3}{2}\right)\right)^2 + \frac{\omega^2\hbar^2\beta_1}{8}\right). \tag{6.14}$$

In the y axis we have the Heisenberg algebra of the harmonic oscillator. With results of section 2 we can reobtain the third-order integral and the cubic algebra [18] of this superintegrable systems. Let us now consider a system of the form given by equation (6.11) with the function $g_1(x)$ satisfying the equation (6.12) and the function $g_2(y)$ satisfying also the equation (6.12) (by replacing respectively x, $f_1, \alpha_1, \beta_1, \omega_1$ and ϵ_1 by y, $f_2, \alpha_2, \beta_2, \omega_2$ and ϵ_2). We impose $m\omega_1 = n\omega_2 = \tilde{\omega}$ with $m, n \in \mathbb{Z}^+$. We thus have by the results of section 6 constructed a new superintegrable systems with higher order integrals. Thus, the method of sections 2 and 6 can be used to generate new superintegrables systems with higher integrals from known one-dimensional quantum systems with ladder operators. These one-dimensional systems could be obtained in the context of SUSYQM or HSQM. This is interesting because it was shown that the search and the classification of systems with the higher order integral of motion is a difficult task [14, 15]. The polynomial algebra can be obtained in the form of a deformed oscillator algebra. The structure function of the system given by equation (6.12) is given by equations (6.3) and (6.4)

$$\begin{aligned}\Phi_{m,n}(x) = \prod_{i=1}^{m} \tilde{\omega}^6\hbar^6 &\left(\frac{E}{2\hbar\tilde{\omega}} + x + u - 1 + \frac{i}{m} - \gamma_{0,1}\right) \\ &\times \left(\frac{E}{2\hbar\tilde{\omega}} + x + u - 1 + \frac{i}{m} - \gamma_{-,1}\right)\left(\frac{E}{2\hbar\tilde{\omega}} + x + u - 1 + \frac{i}{m} - \gamma_{+,1}\right) \\ &\prod_{j=1}^{n}\left(\frac{E}{2\hbar\tilde{\omega}} - x - u + \frac{j}{n} - \gamma_{0,2}\right)\left(\frac{E}{2\hbar\tilde{\omega}} - x - u + \frac{j}{m} - \gamma_{-,2}\right) \\ &\times \left(\frac{E}{2\hbar\tilde{\omega}} - x - u + \frac{j}{n} - \gamma_{+,2}\right),\end{aligned} \tag{6.15}$$

$$\gamma_{0,j} = -\frac{1}{3m}(-3 + \alpha_j - \epsilon_j), \qquad \gamma_{\pm,j} = \frac{\hbar\omega_1}{12m}(-6 + 2\alpha_j \pm 3i\sqrt{2\beta_j} + 16\epsilon_j). \tag{6.16}$$

We obtain the finite-dimensional unitary representations and the corresponding energy spectrum from equation (4.6). Let us present one of the nine solutions with $p = 1, 2, \ldots, m$, $q = 1, 2, \ldots, k$ and $N \in \mathbb{N}$:

$$u_1 = \frac{-E}{2\hbar\tilde{\omega}} + 1 - \frac{p}{m} + \gamma_{0,1}, \qquad E_1 = \hbar\tilde{\omega}\left(N + 2 - \frac{p}{m} - \frac{q}{n} + \gamma_1 + \gamma_{0,2}\right), \tag{6.17}$$

$$\Phi_1 = \prod_i^m \prod_j^n \tilde{\omega}^6 \hbar^6 \left(x + \frac{i-p}{m}\right)\left(x + \frac{i-p}{m} + \gamma_{0,1} - \gamma_{-,1}\right)\left(x + \frac{i-p}{m} + \gamma_{0,1} - \gamma_{+,1}\right)$$
$$\times \left(N + 1 - x + \frac{j-q}{n}\right)\left(N + 1 - x + \frac{j-q}{n} + \gamma_{0,2} - \gamma_{-,2}\right)$$
$$\times \left(N + 1 - x + \frac{j-q}{n} + \gamma_{0,2} - \gamma_{+,2}\right). \quad (6.18)$$

7. Conclusion

In this paper, we constructed the integrals of motion and the polynomial algebra for two-dimensional Hamiltonians of the form given by equation (2.1) from the ladder operators of the one-dimensional Hamiltonians H_x and H_y. The polynomial algebra for potentials 5 and 6 are respectively seventh-order and quintic algebras.

We have also studied the realization in terms of deformed oscillator algebras of a class of the polynomial algebras of the seventh order. These results allowed us to obtain the structure function for potentials 5 and 6 and to obtain unitary representations with their corresponding energy spectrum. We studied with this method a family of caged anisotropic oscillator and a new superintegrable system involving the fourth Painlevé transcendent. We found the polynomial algebra, the finite-dimensional unitary representations and the corresponding energy spectrum.

Superintegrable systems with third-order or higher order integrals do not coincide [14, 15]. However, the method discussed in sections 2 and 6 of this paper could be discussed in the context of classical mechanics in terms of the Poisson bracket and polynomial Poisson algebras. These results could also be generalized for systems that separate in Cartesian coordinates in higher dimensions.

The classification of systems with ladder operators or higher order supersymmetry is important and could also allow us to find new superintegrable systems. A classification of systems in E_2 with second-order ladder operators and the relation with Smorodinsky–Winternitz systems [6] was discussed in [52]. In context of supersymmetry, a class of Hamiltonians with third-order ladder operators were discussed in [18, 33] and fourth-order ladder operators in [34]. These systems involve respectively the fourth and the fifth Painlevé transcendent. Superintegrability and their polynomial algebras, ladder operators and supersymmetric quantum mechanics appear to be closely connected and the study of these connections is important.

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